Math 2050, note

Theorem 0.1 (Bolzano-Weierstrass Theorem)**.** *Suppose $\{x_n\}$ is a bounded sequence, then there is a convergent subsequence.*

We will give an alternative proof which is different from that in textbook.

Proof. By boundedness, there is a, b such that for all n,

$$a \le x_n \le b.$$

For $k = 0$, we denote $I_0 = [a, b]$, $a_0 = a$ and $b_0 = b$. Suppose $[a, \dfrac{a_0 + b_0}{2}]$ contains infinity many x_k, then we choose $a_1 = a_0$, $b_1 = \frac{a_0+b_0}{2}$ otherwise we choose $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$. Then we define $I_1 = [a_1, b_1]$ and pick $x_{n_1} \in I_1$. This is possible since I_1 contains infinity many elements.

We repeat the same step to obtain a sequence of I_k so that I_k is a sequence of closed, bounded and nested sequence. Moreover, there is $x_{n_k} \in I_k$ and

$$|I_k| = \frac{b-a}{2^k}.$$

By nested interval theorem, we have $\eta \in \cap_{k=1}^{\infty} I_k$. Therefore,

$$|\eta - x_{n_k}| \le |I_k| = \frac{b-a}{2^k}$$

which implies $x_{n_k} \to \eta$ as $k \to +\infty$. □

Proposition 0.1. *Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of real number which is divergent and bounded, then there exists $\{x_{n_m}\}_{m=1}^{\infty}$ and $\{x_{k_m}\}_{m=1}^{\infty}$ which is convergent so that* $\lim_{m\to+\infty} x_{n_m} = \bar{x} \neq \tilde{x} = \lim_{m\to+\infty} x_{k_m}$.

Proof. Since the sequence is not convergent, the following is true in general (regardless of bounded or not):

For any $x \in \mathbb{R}$, there exists $\varepsilon_x > 0$ such that for all $n \in \mathbb{N}$, there is $m > n$ satisfying

$$|x_m - x| \ge \varepsilon_x > 0.$$

Now, we can apply the trick we learn in the last lecture to construct a sub-sequence $\{x_{n_m}\}_{m=1}^{\infty}$ so that for all $m \in \mathbb{N}$,

$$|x_{n_m} - x| \ge \varepsilon_x > 0.$$

Since we know in addition the sequence is bounded, so does all its sub-sequence. (Keeping in mind the very basic example $x_n = (-1)^n$!)

Take x to be 1 (randomly), then we can find a sub-sequence $\{x_{n_m}\}_{m=1}^{\infty}$ so that for all $m \in \mathbb{N}$,

$$|x_{n_m} - 1| \geq \varepsilon_1 > 0.$$

As $\{x_{n_m}\}_{m=1}^{\infty}$ is bounded, it follows from Bolzano-Weierstrass Theorem that there exists a convergent sub-sequence $\{x_{n_{m_j}}\}_{j=1}^{\infty}$ of $\{x_{n_m}\}_{m=1}^{\infty}$ so that $\lim_{j\to+\infty} x_{n_{m_j}} = \bar{x}$ for some $\bar{x} \in \mathbb{R}$. Since the sequence is convergent, we might pass the above (non-strict) inequality to limit and hence $\bar{x} \neq 1$.

(keeping in mind, we have to extract sub-sequence again. Consider the example: $x_n = 0$ if $n = 3k$, $x_n = 1$ if $n = 3k+1$, $x_n = 2$ if $n = 3k+2$, then it is bounded divergent sequence. And if we fix $x = 1$, then we might obtain subsequence in form of mixture of $3k, 3k' + 2$ which is still divergent)

So now $\bar{x} \in \mathbb{R}$ and we know that x_n does not converge to $\bar{x}$. Hence, we can find a sub-sequence $\{x_{k_m}\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that for all $m \in \mathbb{N}$,

$$|x_{k_m} - \bar{x}| \geq \varepsilon_{\bar{x}} > 0.$$

Again, as $\{x_{k_m}\}_{m=1}^{\infty}$ is bounded, it follows from Bolzano-Weierstrass Theorem that there exists a convergent sub-sequence $\{x_{k_{m_j}}\}_{j=1}^{\infty}$ of $\{x_{k_m}\}_{m=1}^{\infty}$ so that $\lim_{j\to+\infty} x_{n_{k_j}} = \tilde{x}$ for some $\tilde{x} \in \mathbb{R}$. We can again pass the inequality to limit to conclude $\tilde{x} \neq \bar{x}$. Since both $\{x_{n_m}\}_{m=1}^{\infty}$ and $\{x_{k_m}\}_{m=1}^{\infty}$ are sub-sequence of $\{x_n\}_{n=1}^{\infty}$, we are done.

$\square$